

# بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ



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# ① Limits:

النهايات

## ① البسيط المباشر

$$EX: \lim_{x \rightarrow 2} \frac{3x+1}{x+3} = \frac{6+1}{2+3} = \frac{7}{5} \checkmark$$

$$EX: \lim_{x \rightarrow 3} \frac{x^2-9}{x+2} = \frac{9-9}{3+2} = \frac{0}{5} = 0 \checkmark$$

ك إذا كان البسيط المباشر يعطي  $\frac{صفر}{صفر}$  هنالك حالتان

② قسمة - قسمة - قسمة

$$EX: \lim_{x \rightarrow 3} \frac{x^2-9}{x-3} \quad \left( \frac{9-9}{3-3} = \frac{0}{0} \right)$$

$$= \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{(x-3)} = 3+3 = 6$$

$$\underline{EX}: \lim_{x \rightarrow 2} \frac{x^2-4}{x-2} = \left( \frac{0}{0} \right)$$

$$= \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{x-2} = 2+2 = 4$$

$$EX: \lim_{x \rightarrow 3} \frac{x^2-x-6}{x^2-8x+15} \quad \left( \frac{0}{0} \right)$$

$$= \lim_{x \rightarrow 3} \frac{(x+2)(x-3)}{(x-3)(x-5)} = \frac{3+2}{3-5} = \frac{5}{-2}$$

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$$EX: \lim_{x \rightarrow 2} \frac{x^2 + 2x - 8}{x^2 - 4}$$

$$= \lim_{x \rightarrow 2} \frac{(x-2)(x+4)}{(x-2)(x+2)} = \frac{2+4}{2+2} = \frac{6}{4} = \frac{3}{2}$$

کے لیے خوف الیبت :

$$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$$

$$EX: x^3 - 8 = x^3 - 2^3 = (x-2)(x^2 + 2x + 4)$$

$$EX: x^3 - 27 = x^3 - 3^3 = (x-3)(x^2 + 3x + 9)$$

$$EX: \lim_{x \rightarrow 2} \frac{x^2 - 4}{x^3 - 8} = \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{(x-2)(x^2 + 2x + 4)}$$

$$= \frac{2+2}{4+4+4} = \frac{4}{12} = \frac{1}{3}$$

$$EX: \lim_{x \rightarrow 3} \frac{x^2 - 9}{x^3 - 27} = \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{(x-3)(x^2 + 3x + 9)}$$

$$\frac{3+3}{9+9+9} = \frac{6}{27} = \frac{2}{9}$$

$$EX: \lim_{x \rightarrow 9} \frac{x - 9}{\sqrt{x} - 3} = \lim_{x \rightarrow 9} \frac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{(\sqrt{x} - 3)}$$

$$= \sqrt{9} + 3 = 3 + 3 = 6$$

$$EX: \lim_{x \rightarrow 25} \frac{\sqrt{x} - 5}{x - 25} = \lim_{x \rightarrow 25} \frac{\sqrt{x} - 5}{(\sqrt{x} - 5)(\sqrt{x} + 5)}$$

$$= \frac{1}{\sqrt{25} + 5} = \frac{1}{5 + 5} = \frac{1}{10}$$

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(ب) الصرب بالرافقا

في حالة وجود جذر تربيعي جديد

$$\boxed{(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = a - b}$$

EX:  $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

$$= \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} \cdot \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$$

$$= \lim_{x \rightarrow 0} \frac{(1+x) - (1-x)}{x(\sqrt{1+x} + \sqrt{1-x})}$$

$$= \lim_{x \rightarrow 0} \frac{2x}{x(\sqrt{1+x} + \sqrt{1-x})} = \frac{2}{\sqrt{1+0} + \sqrt{1-0}}$$

$$= \frac{2}{1+1} = \frac{2}{2} = 1$$

EX:  $\lim_{x \rightarrow 3} \frac{\sqrt{x+6} - 3}{x-3}$

$$= \lim_{x \rightarrow 3} \frac{\sqrt{x+6} - 3}{x-3} \cdot \frac{\sqrt{x+6} + 3}{\sqrt{x+6} + 3}$$

$$= \lim_{x \rightarrow 3} \frac{x+6-9}{(x-3)(\sqrt{x+6} + 3)}$$

$\sqrt{9}$



②

$$= \lim_{x \rightarrow 3} \frac{x-3}{(x-3)(\sqrt{x+6}+3)} = \frac{1}{\sqrt{3+6}+3} = \frac{1}{\sqrt{9}+3}$$

$$= \frac{1}{3+3} = \frac{1}{6}$$

③ إذا كان  $x \rightarrow \infty$  : قسم لبط، انقسم على  $x$  باليد أيسر

EX:  $\lim_{x \rightarrow \infty} \frac{3x^2 + 2x - 1}{5x^2 + 4}$

$$= \lim_{x \rightarrow \infty} \frac{\frac{3x^2}{x^2} + \frac{2x}{x^2} - \frac{1}{x^2}}{\frac{5x^2}{x^2} + \frac{4}{x^2}} = \lim_{x \rightarrow \infty} \frac{3 + \frac{2}{x} - \frac{1}{x^2}}{5 + \frac{4}{x^2}}$$

$$= \frac{3 + \frac{2}{\infty} - \frac{1}{\infty^2}}{5 + \frac{4}{\infty^2}} = \frac{3 + 0 - 0}{5 + 0} = \frac{3}{5}$$

EX:  $\lim_{x \rightarrow \infty} \frac{2x^2 - 1}{5x^3 - 2x + 4}$

$$= \lim_{x \rightarrow \infty} \frac{\frac{2x^2}{x^3} - \frac{1}{x^3}}{\frac{5x^3}{x^3} - \frac{2x}{x^3} + \frac{4}{x^3}} = \lim_{x \rightarrow \infty} \frac{\frac{2}{x} - \frac{1}{x^3}}{5 - \frac{2}{x^2} + \frac{4}{x^3}}$$

$$= \frac{\frac{2}{\infty} - \frac{1}{\infty^3}}{5 - \frac{2}{\infty^2} + \frac{4}{\infty^3}} = \frac{0 - 0}{5 - 0 + 0}$$

$$= \frac{0}{5} = 0$$

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$$EX: \lim_{x \rightarrow \infty} \frac{2x^3 + 2}{3x^2 - 5x + 7}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{2x^3}{x^3} + \frac{2}{x^3}}{\frac{3x^2}{x^3} - \frac{5x}{x^3} + \frac{7}{x^3}}$$

$$= \lim_{x \rightarrow \infty} \frac{2 + \frac{2}{x^3}}{\frac{3}{x} - \frac{5}{x^2} + \frac{7}{x^3}} = \frac{2 + \frac{2}{\infty^3}}{\frac{3}{\infty} - \frac{5}{\infty^2} + \frac{7}{\infty^3}}$$

$$= \frac{2 + 0}{0 - 0 + 0} = \frac{2}{0} = \infty$$

$\frac{0}{a} = 0$	$\frac{a}{0} = \infty$	$\frac{a}{\infty} = 0$
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نظري \*

$$\lim_{x \rightarrow \infty} [1 \oplus F(x)]^{g(x)} = e^{\lim_{x \rightarrow \infty} F(x) \cdot g(x)}$$

$$EX: \lim_{x \rightarrow \infty} \left[1 + \frac{1}{x}\right]^{3x} = e^{\lim_{x \rightarrow \infty} \left(\frac{1}{x}\right)(3x)}$$

$$= e^{\lim_{x \rightarrow \infty} 3} = e^3$$

$$\lim_{x \rightarrow a} C = C$$

⑦

$$\text{EX} \quad \lim_{x \rightarrow \infty} \left[ 1 + \frac{2}{x^2} \right]^{\frac{x^2}{3}}$$

$$= \lim_{x \rightarrow \infty} \left( \frac{2}{x^2} \right) \left( \frac{x^2}{3} \right) = e^{\frac{2}{3}}$$

$$\text{EX:} \quad \lim_{x \rightarrow \infty} \left[ 1 - \frac{2}{x} \right]^{\frac{x}{3}}$$

$$= \lim_{x \rightarrow \infty} \left[ 1 + \frac{-2}{x} \right]^{\frac{x}{3}} = \lim_{x \rightarrow \infty} \left( \frac{-2}{x} \right) \left( \frac{x}{3} \right) = e^{-\frac{2}{3}}$$

$$\underline{\text{EX:}} \quad \lim_{x \rightarrow \infty} \left[ \frac{x+4}{x+2} \right]^{3x}$$

$$= \lim_{x \rightarrow \infty} \left[ \frac{\frac{x}{x} + \frac{4}{x}}{\frac{x}{x} + \frac{2}{x}} \right]^{3x} = \lim_{x \rightarrow \infty} \left[ \frac{1 + \frac{4}{x}}{1 + \frac{2}{x}} \right]^{3x}$$

$$= \lim_{x \rightarrow \infty} \frac{\text{graph of } y = 1 + \frac{4}{x}}{\text{graph of } y = 1 + \frac{2}{x}}$$


$$\frac{\lim_{x \rightarrow \infty} \left( 1 + \frac{4}{x} \right)^{3x}}{\lim_{x \rightarrow \infty} \left( 1 + \frac{2}{x} \right)^{3x}} = \frac{\lim_{x \rightarrow \infty} \left( \frac{4}{x} \right) (3x)}{\lim_{x \rightarrow \infty} \left( \frac{2}{x} \right) (3x)}$$

④

$$= \frac{e^{12}}{e^6} = e^{12-6} = e^6$$

تدريب:

Find  $\lim_{x \rightarrow 2} \frac{\frac{1}{x} - \frac{1}{2}}{x-2}$

$$= \lim_{x \rightarrow 2} \frac{\frac{2-x}{2x}}{x-2} = \lim_{x \rightarrow 2} \frac{\frac{-(x-2)}{2x}}{x-2}$$

$$= \frac{-1}{2(2)} = \underline{\underline{-\frac{1}{4}}}$$

⑤ نهاية الدالة جيبية:

①  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

②  $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$

$$\lim_{x \rightarrow 0} \frac{\sin ax}{ax} = 1$$

Ex:

$$\lim_{x \rightarrow 0} \frac{\sin 3x}{2x} = \lim_{x \rightarrow 0} \frac{\frac{3}{2} \sin 3x}{3x}$$

$$= \frac{3}{2} \lim_{x \rightarrow 0} \frac{\sin 3x}{3x}$$

$$= \frac{3}{2} (1) = \frac{3}{2}$$



①

$$\begin{aligned} \text{EX: } \lim_{x \rightarrow 0} \frac{\tan 2x}{5x} &= \lim_{x \rightarrow 0} \frac{2 \tan 2x}{5 \cdot 2x} \\ &= \frac{2}{5} \lim_{x \rightarrow 0} \frac{\tan 2x}{2x} = \frac{2}{5} (1) = \frac{2}{5} \end{aligned}$$

$$\begin{aligned} \underline{\text{EX:}} \quad \lim_{x \rightarrow 0} \frac{\sin 3x}{\tan 5x} &= \lim_{x \rightarrow 0} \frac{\frac{\sin 3x}{x}}{\frac{\tan 5x}{x}} \\ &= \frac{\lim_{x \rightarrow 0} 3 \frac{\sin 3x}{3x}}{\lim_{x \rightarrow 0} 5 \frac{\tan 5x}{5x}} = \frac{3 \lim_{x \rightarrow 0} \frac{\sin 3x}{3x}}{5 \lim_{x \rightarrow 0} \frac{\tan 5x}{5x}} \\ &= \frac{3(1)}{5(1)} = \frac{3}{5} \end{aligned}$$

$$\begin{aligned} \underline{\text{EX:}} \quad \lim_{x \rightarrow 0} \frac{x}{\sin x} &= \lim_{x \rightarrow 0} \frac{\frac{x}{x}}{\frac{\sin x}{x}} \\ &= \frac{\lim_{x \rightarrow 0} 1}{\lim_{x \rightarrow 0} \frac{\sin x}{x}} = \frac{1}{1} = 1 \end{aligned}$$

= مطلوب

①  $1 - \cos 2x = 2 \sin^2 x$

②  $1 - \cos x = 2 \sin^2 \frac{x}{2}$

⑨ \* EX: Find  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$

$$= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x^2} = 2 \lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{x} \cdot \frac{\sin \frac{x}{2}}{x}$$

$$= 2 \lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}} \cdot \lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}}$$

$$= 2 \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) = \frac{1}{2}$$

Sandwich

تقریباً ساندویچی

if  $F(x) \leq g(x) \leq h(x)$

and if  $\lim_{x \rightarrow a} F(x) = L = \lim_{x \rightarrow a} h(x)$

then  $\lim_{x \rightarrow a} g(x) = L$

\*

$$-1 \leq \sin x \leq 1$$

EX: Find  $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}$

Solution

$$-1 \leq \sin \frac{1}{x} \leq 1$$

بالحدود  $x^2$  :

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$$

(1.)

$$\lim_{x \rightarrow 0} -x^2 = -0^2 = 0$$

$$\lim_{x \rightarrow 0} x^2 = 0^2 = 0$$

$$\therefore \lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$$

EX: Find  $\lim_{x \rightarrow 0} \frac{1}{x^2} \sin x^2$

Solution

$$-1 \leq \sin x^2 \leq 1$$

$\frac{1}{x^2}$  is variable

$$\Rightarrow -\frac{1}{x^2} \leq \frac{1}{x^2} \sin x^2 \leq \frac{1}{x^2}$$

$$\lim_{x \rightarrow 0} -\frac{1}{x^2} = -\frac{1}{0} = \infty$$

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \frac{1}{0} = \infty$$

$$\therefore \lim_{x \rightarrow 0} \frac{1}{x^2} \sin x^2 = \infty$$

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الانضال .. Continuous of Function ::

① النهاية ليمت وليست

if  $F(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & x > 2 \\ x + 2 & x < 2 \end{cases}$

Find  $\lim_{x \rightarrow 2} F(x)$

Solution

$$\begin{aligned} \lim_{x \rightarrow 2^+} F(x) &= \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{x-2} \\ &= 2 + 2 = \boxed{4} \end{aligned}$$

$$\lim_{x \rightarrow 2^-} F(x) = \lim_{x \rightarrow 2} (x + 2) = 2 + 2 = \boxed{4}$$

$$\therefore \lim_{x \rightarrow 2^+} F(x) = \lim_{x \rightarrow 2^-} F(x) = 4$$

$$\therefore \lim_{x \rightarrow 2} F(x) = 4$$

EX:  $F(x) = \begin{cases} \frac{\sin x}{3x} & x > 0 \\ \frac{x+2}{3} & x < 0 \end{cases}$

Find  $\lim_{x \rightarrow 0} F(x)$



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Solution

$$\lim_{x \rightarrow 0^+} F(x) = \lim_{x \rightarrow 0} \frac{\sin x}{3x}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x}{3x} = \frac{1}{3} \lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{1}{3}$$

$$\lim_{x \rightarrow 0^-} F(x) = \lim_{x \rightarrow 0} \frac{x+2}{3} = \frac{0+2}{3} = \frac{2}{3}$$

$$\therefore \lim_{x \rightarrow 0^+} F(x) \neq \lim_{x \rightarrow 0^-} F(x)$$

$$\therefore \lim_{x \rightarrow 0} F(x) \text{ Does not exist}$$

∴ دالة لقيمة مختلفة (5)

$$|x-a| = \begin{cases} x-a & x > a \\ -(x-a) & x < a \end{cases}$$

EX: Find  $\lim_{x \rightarrow 2} \frac{|x-2|}{x-2}$

Solution

$$\frac{|x-2|}{x-2} = \begin{cases} \frac{x-2}{x-2} & x > 2 \\ \frac{-(x-2)}{x-2} & x < 2 \end{cases}$$

$$= \begin{cases} 1 & x > 2 \\ -1 & x < 2 \end{cases}$$

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$$\begin{aligned}\lim_{x \rightarrow 2^+} F(x) &= \lim_{x \rightarrow 2} 1 = 1 \\ \lim_{x \rightarrow 2^-} F(x) &= \lim_{x \rightarrow 2} -1 = -1\end{aligned} \quad \left. \begin{array}{l} \text{من اليمين} \\ \text{من اليسار} \end{array} \right\}$$

$$\therefore \lim_{x \rightarrow 2^+} F(x) \neq \lim_{x \rightarrow 2^-} F(x)$$

$\therefore \lim_{x \rightarrow 2} F(x)$  does not exist

٢٠ الفصل

if  $F(x)$  is a function then

$F(x)$  is continuous at  $x=a$  if

$$\lim_{x \rightarrow a^+} F(x) = \lim_{x \rightarrow a^-} F(x) = F(a)$$

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EX:

Discuss if  $F(x)$  is continuous  
at  $x=0$  if

$$F(x) = \begin{cases} \frac{1 - \cos x}{x^2} & x \neq 0 \\ \frac{1}{2} & x = 0 \end{cases}$$

(2)

Solution

$$\lim_{x \rightarrow 0} F(x) = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x^2}$$

$$= 2 \lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{x} \cdot \frac{\sin \frac{x}{2}}{x}$$

$$= 2 \lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{x} \cdot \lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{x}$$

$$= 2 \lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}} \cdot \lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}} = 2 \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) = \frac{1}{2}$$

$$F(0) = \frac{1}{2}$$

$$\therefore \lim_{x \rightarrow 0} F(x) = F(0)$$

$\therefore F(x)$  is continuous

Ex: DISCUS .....

$$F(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & x > 3 \\ 6 & x = 3 \\ x + 3 & x < 3 \end{cases}$$

at  $x = 3$

(10)

Solution

$$\lim_{x \rightarrow 3^+} F(x) = \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{x-3} = 6$$

$$\lim_{x \rightarrow 3^-} F(x) = \lim_{x \rightarrow 3} (x+3) = 3+3 = 6$$

$$F(3) = 6$$

$$\therefore \lim_{x \rightarrow 3^+} F(x) = \lim_{x \rightarrow 3^-} F(x) = F(3)$$

$\therefore F(x)$  is continuous

Ex:

$$F(x) = \frac{|x-4|}{x-4} \quad \text{at } x=4$$

Solution

$$F(x) = \frac{|x-4|}{x-4} = \begin{cases} \frac{x-4}{x-4} & x \geq 4 \\ \frac{-(x-4)}{x-4} & x < 4 \end{cases} = \begin{cases} 1 & x \geq 4 \\ -1 & x < 4 \end{cases}$$

$$\lim_{x \rightarrow 4^+} F(x) = \lim_{x \rightarrow 4} 1 = 1$$

$$\lim_{x \rightarrow 4^-} F(x) = \lim_{x \rightarrow 4} -1 = -1$$

}  $\neq$

$\therefore F(x)$  is not continuous